# Solitary wave dynamics in generalized Hertz chains: An improved solution of the equation of motion

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The equation of motion for a bead in a chain of uncompressed elastic beads in contact that interact via the potential $V(\delta)\sim\delta^n$, $n>2$, $\delta$ being overlap, supports solitary waves and does not accommodate sound propagation [V. Nesterenko, J. Appl. Mech. Tech. Phys. **5**, 733 (1983)]. We present an iteratively exact solution to describe the solitary wave as a function of material parameters and a universal, infinite set of coefficients, which depend only on *n*. We compute any arbitrary number of coefficients to desired accuracy and show that only the first few coefficients of our solution significantly improves upon Nesterenko's solution. The improved solution is a necessary step to develop a theoretical understanding of the formation of *secondary* solitary waves [M. Manciu, *et al.*, Phys. Rev. E **63**, 011614 (2001)].

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We consider a monodisperse chain of elastic spheres of radius $R$ in which the spheres are barely touching one another, i.e., there is zero loading of the spheres. The spheres repel upon overlap by an amount $\delta_{i,i+1}\equiv 2R-[(z_{i+1}+u_{i+1})-(z_i+u_i)]$, where $z_i$ describes the initial equilibrium position of the grain $i$ in the one-dimensional (1D) system of elastic beads, and $u_i$ the displacement of the same grain from the equilibrium position. Then, according to Hertz law [1] the repulsive potential between two adjacent spheres is given by,

$$V(\delta_{i,i+1})=a|\delta_{r,i+1}|^{5/2}, \qquad \delta\geqslant 0,$$

$$V(\delta_{i,i+1})=0, \qquad \delta<0, \tag{1}$$

where $a=(2/[5D(Y,\sigma)])(R/2)^{1/2}$ and $D(Y,\sigma)=(3/2)[(1-\sigma^2)/Y]$, and $Y$ and $\sigma$ denote the Young's modulus and Poisson's ratio. In this paper, we study the dynamics for the potential in Eq. (1) and for the general case $V(\delta_{i,i+1})=a\delta_{i,i+1}{}^n$, $n>2$ [1,2]. Given the magnitude of *n*, it may be noted that the repulsive potential $V(\delta_{i,i+1})$, as stated in Eq. (1), is intrinsically nonlinear in the sense that it cannot be linearized. This statement implies that sound propagation is not possible in a chain of elastic beads at zero precompression, a phenomenon that has been referred to in the literature as ''sonic vacuum'' [3].

To initiate sound propagation, one must introduce some precompression in the system. The simplest case is uniform precompression, say $\Delta$, effected on every grain. The equation of motion of a bead in the chain (not at the boundaries) then becomes,

$$md^2u_i(t)/dt^2=na\{[\Delta+u_{i-1}(t)-u_i(t)]^{n-1}$$
$$-\{[\Delta+u_i(t)-u_{i+1}(t)]^{n-1}\}, \tag{2}$$

where the right-hand side (RHS) of Eq. (2) can be expanded when $\Delta>0$. Nesterenko [3] showed that if $u_i$ varies slowly in space, i.e., if the long-wavelength limit is invoked, and if $\Delta\to 0$ and $n\ll\infty$, then Eq. (2) can be approximated via a Korteweg–de Vries-type [3,4] equation, which admits a soliton solution for a propagating perturbation in the chain. Nesterenko's analysis has been improved [5], tested numerically [6], and has been experimentally verified [3,7]. It is presently well known that the solitary waves in 1D systems are typically about 3 grain diameters wide [3,5]. High-precision numerical studies indicate that two identical solitons, propagating in opposite directions in a chain with an even number of grains, completely negate one another at the point of crossing but do not completely negate one another in the vicinity of the point of crossing. Therefore, it would be appropriate to characterize these excitations as *solitary waves* rather than solitons.

In fact, the existence of solitary waves in granular media is so robust that even in 3D, when a large area impulse or a low-frequency (below a few kHz) acoustic signal is generated at the surface of a granular bed, the impulse travels as a weakly dispersive energy bundle, a phenomenon that has been simulated and experimentally validated [8].

Given the small size of the solitary waves in 1D and in higher dimensions, it would be desirable to have an exact solution or perhaps a more complete description of the solution to Eq. (2) that would be valid for arbitrary *n* and would be appropriate for the construction of analytic descriptions of processes involving colliding solitary waves and backscattering of solitary waves from interfaces with density contrasts (where long-wavelength approaches are no longer useful). Constructing an exact closed form solution to Eq. (2) remains a challenge. In this paper, we report an exact solution that can be generated using a hybrid technique that exploits numerical analysis and an analytic form to describe the solitary waves. As we shall see, the solution, does not require the long-wavelength approximation, directly solves Eq. (2) for $\Delta=0$, and may be constructed to desired accuracy provided the material parameters and *n* are known. This solution significantly improves upon Nesterenko's original solution [3]. The present paper also allows us to revisit the Fermi-Pasta-Ulam (FPU) problem [9]. We show that our solution may be interpreted as a solution to the FPU problem in an appropriate asymptotic limit.

We start by assuming that a solitary wave is propagating through the system. The displacement of individual grains from equilibrium position $u_i(t)$ are continuous functions of time but are defined only at discrete positions $z_i$. Since the solitary wave is nondispersive, we may assume that this displacement can be obtained from a wave-type continuous function of both space and time, from the relation

$$u_i(t)=u(z_i,t)=u(z_i-ct)=u(\alpha),\quad \text{with}\quad \alpha=z-ct, \tag{3}$$

where $c$ is the constant velocity of the solitary wave. It may be noted that in spite of the continuum approximation in space introduced above, the analytic solution to Eq. (2) developed here is in impressive agreement with the numerically generated solution of the discrete system.

Our exhaustive numerical studies on Eq. (2) and also other work [3] indicate that, for a given $n$, the shape of the solitary wave in space does not depend on the solitary wave amplitude. This implies that the function $u$ is described by $u(\alpha)=A\psi_n(\alpha)$, where $A$ represents the amplitude of the solitary wave and $A=u(-\infty)-u(+\infty)=1$. The quantity $\psi_n(\alpha)$ is an unknown generic function that describes the shape of the solitary wave and is expected to depend upon the index $n$, which controls the stiffness of the potential. Because the solitary wave at any time is localized in space, $\psi_n(\alpha)$ should be necessarily zero for $\alpha\to\infty$ ($z\to\infty$ for finite $t$, which represent a region that the solitary wave is yet to reach) and 1 for $\alpha\to-\infty$ (where grains have attained a new equilibrium position after the passage of the compression produced by the tsunamilike or kink solitary wave). A function that respects this boundary condition and may only take intermediate values between 0 and 1, may be always written as

$$\psi_n(\alpha)=1/\{1+\exp[f_n(\alpha)]\},$$
$$\text{with}\quad f_n(\alpha)=\ln[1/\psi_n(\alpha)-1]. \tag{4}$$

With this notation, the solitary wave function becomes:

$$u(\alpha)=(A/2)[\varphi_n(\alpha)+1],\quad \text{with}\quad \varphi_n(\alpha)=-\tanh[f_n(\alpha)/2]. \tag{5}$$

One can see from Eq. (3) that $du/dz|_t=-(1/c)du/dt|_z$. Substituting Eq. (5) into Eq. (2), we get, for $t=0$

$$\begin{aligned}(mc^2/na)(A/2)^{n-2}&=[\{\varphi_n(z-2R)-\varphi_n(z)\}^{n-1}-\{\varphi_n(z)\\&\quad-\varphi_n(z+2R)\}^{n-1}]/[d^2\varphi_n/dz^2]\\&\equiv C_0(n),\end{aligned} \tag{6}$$

where the left-hand side (LHS) is independent of $z$ and the RHS is independent of $m$, $a$, and $A$. Thus, $C_0$ should be independent of $z$, $m$, $a$, and $A$, which means that $C_0$ is a constant that depends only on $n$.

The assumption that Eq. (2) admits a solitary wave solution along with Eqs. (4)–(6) imply that $\varphi_n(z)$ is antisymmetric with respect to $z=0$ or an arbitrary constant, which is the center of solitary wave (recall that $t$ was set to zero). This fact, combined with the asymptotic limits for $\psi_n(z)$ and Eq. (4) indicates that

$$f_n(z)=\sum_{q=0}^{\infty} C_{2q+1}(n)z^{2q+1}. \tag{7}$$

Since the function $\varphi_n(z)$ is independent of all quantities except $n$, knowledge of the coefficients $C_0$, $C_1$, $C_3$, $C_5,\ldots,$ will completely solve the problem of pulse propagation for any system supporting this type of solitary wave. In the absence of a simple analytical approach for inferring $C_0$ and $C_{2q+1}$, one must resort to numerical methods for computing these coefficients.

We now present a hybrid analytical-cum-simulational approach that allows computation of $C_{2q+1}$ to any desired accuracy via an iterative approach.

Recall that Eq. (6) implies that

$$c=\{naC_0(n)/m\}^{1/2}(A/2)^{(n-2)/2}=d_0A^{(n-2)/2}, \tag{8}$$

which implies that the propagation velocity of the solitary wave scales with its amplitude except when $n\to2$, where $c$ becomes independent of amplitude, as expected.

As stated earlier, one obtains a solitary wave by numerical integration of Eq. (2) with $\Delta=0$ [6]. By generating solitary waves with different amplitudes and measuring their velocity, one can compute $d_0$ and hence, $C_0$ from Eq. (8). Note that Eq. (8) does not depend on the other $C$'s.

A derivative of Eq. (5) with respect to $t$ (recognizing that $\varphi_n(z)=\varphi_n(\alpha)=\varphi_n(z-ct)$) will yield, after using Eq. (8), the maximum velocity of the grain for $\alpha=0$

$$\begin{aligned}v_{\max}&=du(z-ct)/dt|_{\alpha=0}=-c\,du(\alpha)/dz|_{\alpha=0}\\&=-(cA/2)d\varphi_n(z)/dz|_{\alpha=0}\\&=(naC_0/m)^{1/2}(C_1/2)(A/2)^{n/2}=d_1A^{n/2}.\end{aligned} \tag{9}$$

Again, the maximum velocity of the grain during the propagation of solitary waves with different amplitudes can be computed, which will offer the scaling coefficient $d_1$ and hence, $C_1$, via Eq. (9).

The antisymmetry of the solution in Eq. (7) implies that even derivatives vanish at the center of the solitary wave $\alpha=0$. However, we find,

$$\begin{aligned}d^3u/dz^3|_{\alpha=0}&=(naC_0/m)^{3/2}(3C_3-C_1^3/4)(A/2)^{(3n-4)/2}\\&=d_3A^{(3n-4)/2},\end{aligned} \tag{10}$$

$$\begin{aligned}d^5u/dz^5|_{\alpha=0}&=(naC_0/m)^{5/2}(60C_5-15C_1^2C_3+C_1^5/2)\\&\quad\times(A/2)^{(5n-8)/2}=d_5A^{(5n-8)/2}.\end{aligned} \tag{11}$$

In Fig. 1 we present numerically obtained data for $c$, $v_{\max}=du/dt|_{\alpha=0}$, $d^3u/dt^3|_{\alpha=0}$, and $d^5u/dt^5|_{\alpha=0}$ versus $A$ for the case $n=2.5$. We directly obtain $d_0=0.7791$, $d_1$

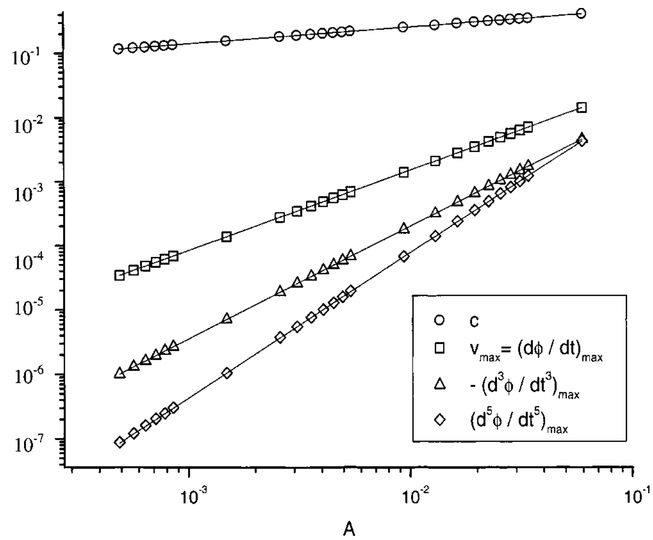


FIG. 1. Velocity of the solitary wave and odd derivatives with respect to time of the grain displacement are shown at the symmetry point of the solitary wave for different amplitudes of the solitary wave. All the units are arbitrary. Linear regressions of the numerical data provide power-law coefficients that are within 0.01% to the values predicted by Eqs. (9)–(12) and the unknown set of $d$ coefficients.

$=0.4666$, $d_3=0.6221$, and $d_5=2.3957$ by linear regression of the data. Using these values in Eqs. (8)–(11) yields $C_0=0.8585$, $C_1=2.3953$, $C_3=0.2685$, and $C_5=0.006\,13$.

Figure 2 presents the numerically simulated data for $\varphi_n$ and its time derivatives, $d\varphi_n/dt$, $d^2\varphi_n/dt^2$, $d^3\varphi_n/dt^3$, $d^4\varphi_n/dt^4$, and $d^5\varphi_n/dt^5$ as functions of $z$ (circles) and comparison with the solutions generated by employing the above-calculated coefficients in Eqs. (5) and (7) for $n=5/2$. The coordinate $z$ is measured in grain diameters $(2R)$ and the units for the derivatives of $\varphi_n$ with respect to $t$ are arbitrary. The higher-order coefficients are neglected. The analysis can be extended to obtain higher-order coefficients, if necessary. For cases with $n \leqslant 5$, the first few coefficients are given in Table I. These coefficients give excellent agreement with numerical solutions (Fig. 2).

TABLE I. Values of $C_{2q+1}(n)$ in [Eq. (7)], for different values of $n$, obtained with the method described in text.

| $n$ | $C_0$ | $C_1$ | $C_3$ | $C_5$ |
|---|---|---|---|---|
| 2.2 | 0.8709(6) | 1.643(7) | 0.082 23(9) | 0.000 325 7(8) |
| 2.35 | 0.6908(5) | 2.3171(6) | 0.2364(4) | 0.003 407(4) |
| 2.5 | 0.858 52(9) | 2.3953(6) | 0.268 52(9) | 0.006 134(7) |
| 3.0 | 0.9445(1) | 3.0168(2) | 0.5971(0) | 0.0376(4) |
| 4.0 | 1.3323(7) | 3.5646(1) | 1.331(4) | 0.0676(3) |
| 5.0 | 2.0517(4) | 3.790 01(3) | 2.177(5) | 0.0665(1) |

In order to quantify the improvement achieved by the present solution to Eq. (2) compared to Nesterenko's solution, in the upper panel of Fig. 3 we plotted the LHS and the RHS of Eq. (2) as obtained via Nesterenko's solution [3] [(c) and (d)], and that obtained using Eq. (7) with the appropriate coefficients provided by the hybrid numerical-analytical approach described (a) and (b). The ratio of the right and left term of Eq. (2) is also offered in Fig. 3 (lower panel), and demonstrates that the present solution [case (e)] is a significant improvement over case (f), which shows Nesterenko's solution.

It turns out that the celebrated Fermi-Pasta-Ulam problem considered a potential that can be written as

$$V(\delta_{i,i+1}) = k\,\delta_{i,i+1}^2 + a|\delta_{i,i+1}|^n \qquad (12)$$

where $\delta_{i,i+1}$ represent the compression and extension of a spring connecting particles $i$, $i+1$ in a chain and the absolute value of the nonlinear term is used to define a stable equilibrium for particles for all values of $a$, $k$, and $n$. FPU consid-

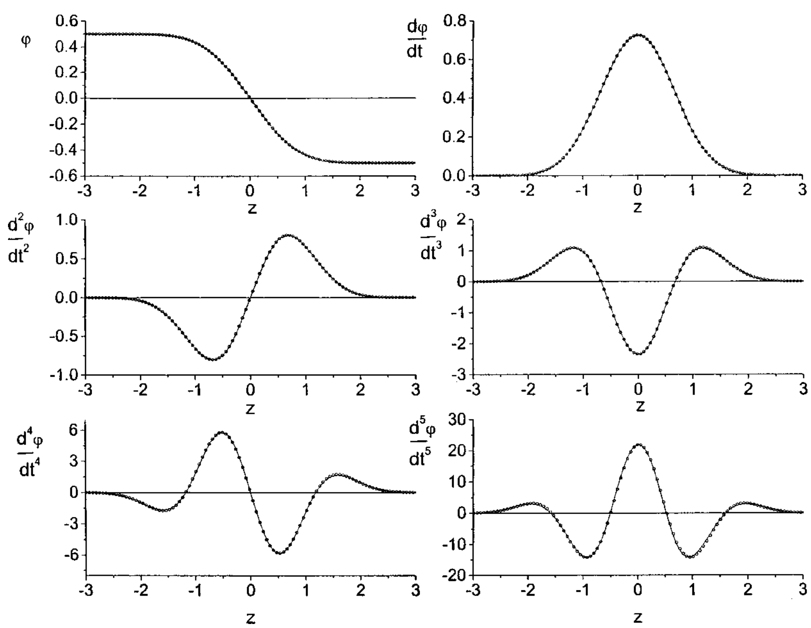


FIG. 2. $\varphi_n(z)$ and higher derivatives, with respect to $t$, for $n=5/2$ (Hertzian chain), as functions of $z$. The coordinate $z$ is measured in grain diameters and $\varphi_n(z)$ has asymptotic values $\pm 1/2$. The derivatives of $\varphi_n$ with respect to time have arbitrary units (since the solitary wave velocity depends on material parameters and $A$), but their shape is unique. Circles represent data obtained via particle dynamics integration of Eq. (2); the continuous line is the analytic solution of Eqs. (5) and (7), with the corresponding coefficients from Table I.

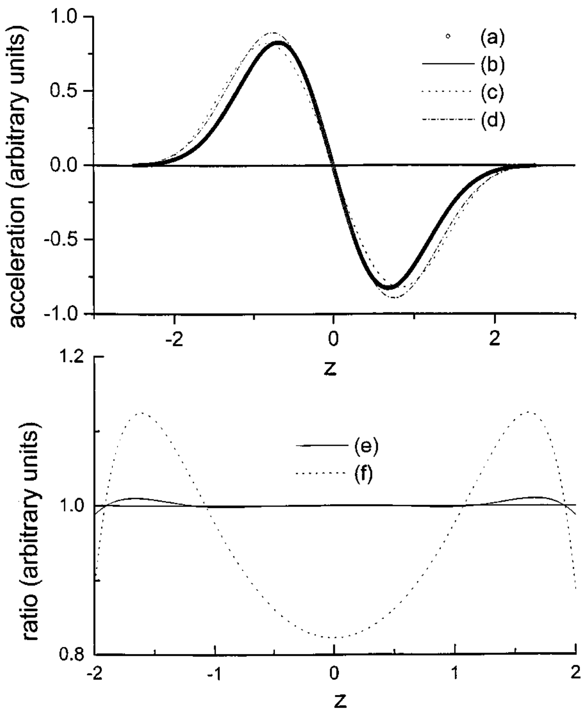


FIG. 3. Comparison between the present solution and the solution obtained by Nesterenko in the long wavelength approximation. In the upper panel, (a) and (b) represent the LHS and RHS, respectively, of Eq. (2), computed with the present solution. (c) and (d) are the corresponding curves computed with Nesterenko's solution [3]. In the lower panel, the ratio of the LHS and RHS of Eq. (2) is computed with the present approach (e) and from Nesterenko's solution (f).

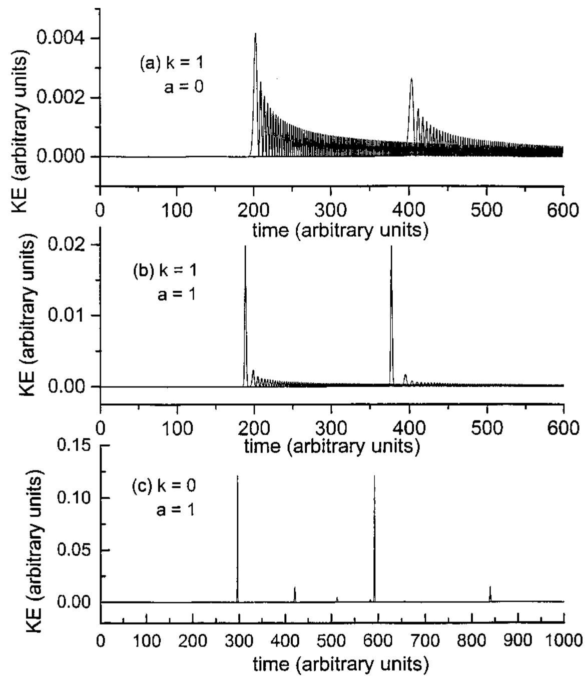


FIG. 4. Kinetic energy for grains 200 and 400 of the chain, as function of time (all the units are arbitrary), when a $\delta$ function perturbation is initiated at the beginning of the chain (grain 1). In the linear case (a), the signal disperses during propagation. The dispersion is weaker when nonlinear terms are present in the potential (b). For purely nonlinear interaction, a perfect solitary wave propagates through the chain (c).

ered $n=3,4$ in their original work [9]. We show calculations to describe the dynamics associated with the propagation of an impulse in the FPU chain for $a=1$, $k=0$; $a=1$, $k=1$, and $a=0$, $k=1$ in Fig. 4. It is obvious looking at Fig. 4 that the FPU system will never equilibrate because it admits Hertz-type solitary waves when $k\rightarrow 0$ [9,10]. Interestingly, our solution to Eq. (2) also serves as a solution to Eq. (12) in the limit of $k\rightarrow 0$.

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